1 Exercise 5.10.5 (a)

Proof:
(1) Show that $\int_{\mathbb{R}} F(x)F(dx) = 1/2$. Actually,

$$\begin{aligned}
\int_{\mathbb{R}} F(x)F(dx) &= \int_{\mathbb{R}} \left[\int_{(-\infty,x]} F(dy)\right] F(dx) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} 1_{(-\infty,x]}(y)F(dy)\right] F(dx) \\
\text{(by Fubini)} &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} 1_{[y,\infty)}(x)F(dx)\right] F(dy) \\
(F \text{ is continuous}) &= \int_{\mathbb{R}} [1 - F(y)]\, F(dy) \\
&= \int_{\mathbb{R}} F(dy) - \int_{\mathbb{R}} F(y)F(dy) \\
&= 1 - \int_{\mathbb{R}} F(x)F(dx) \\
\Longrightarrow \quad & \quad 2\int_{\mathbb{R}} F(x)F(dx) = 1 \\
\Longrightarrow \quad & \quad \int_{\mathbb{R}} F(x)F(dx) = \frac{1}{2}
\end{aligned}$$

Note: An alternative proof is based on Exercise 3.4.5 on page 86.

$$\begin{aligned}
& X \sim F, \text{ which is continuous} \\
\Longrightarrow\ & Y = F(X) \sim \text{Uniform}[0,1] \\
\Longrightarrow\ & \int_{\mathbb{R}} F(x)F(dx) = E\left[F(X)\right] = F[Y] = \frac{1}{2}
\end{aligned}$$

(2) Show that $P[X_1 \le X_2] = 1/2$ if X_1, X_2 iid $\sim F$. Actually,

$$\begin{aligned}
P[X_1 \le X_2] &= \int_{\mathbb{R}^2} 1_{[x_1 \le x_2]}(x_1, x_2)d(F \times F) \\
\text{(by Fubini)} &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} 1_{(-\infty,x_2]}(x_1)F(dx_1)\right] F(dx_2) \\
&= \int_{\mathbb{R}} F(x_2)F(dx_2) \\
\text{(by (1))} &= \frac{1}{2}
\end{aligned}$$

(3) Show that $E\left(F(X_1)\right) = 1/2$. Actually,

$$E\left(F(X_1)\right) = \int_{\mathbb{R}^2} F(x_1)d(F \times F)$$

$$\begin{aligned}
\text{(by Fubini)} \quad &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} F(x_1) F(dx_1) \right] F(dx_2) \\
\text{(by (1))} \quad &= \int_{\mathbb{R}} \frac{1}{2} F(dx_2) \\
&= \frac{1}{2}
\end{aligned}$$

□

2 Exercise 5.10.6 (a) (b) (d)

Proof: First note: since $X \in L_1$, that is, $E(|X|) < \infty$, then $|X| \in L_1$ and $P[|X| = \infty] = 0$. Otherwise, $P[|X| = \infty] > 0$ implies that $E(|X|) = \infty$.

(a) Based on the above, $\lim_{n\to\infty} \downarrow \{|X| > n\} = N$ with $P(N) = 0$. Hence we have

$$\lim_{n\to\infty} X I_{\{|X|>n\}} = 0 \quad \text{a.s.}$$

By Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{\{|X|>n\}} X dP = \lim_{n\to\infty} \int_{\Omega} X I_{\{|X|>n\}} dP = 0.$$

(b) Decompose

$$\int_{A_n} |X| dP = \int_{A_n \{|X| \le M\}} |X| dP + \int_{A_n \{|X|>M\}} |X| dP = I_1 + I_2.$$

For any fixed $\varepsilon > 0$, by (a), we can choose M large enough such that $I_2 < \varepsilon$. With this fixed M, we look at I_2. By monotonicity of integration, we have $I_2 < MP\{A_n\}$. Since $P\{A_n\} \to 0$, we know that for large enough n, we have $P\{A_n\} < \frac{\varepsilon}{M}$. This implied that when n large enough, we have $I_2 < \varepsilon$. Summing up, we conclude that for n large enough, we have

$$\int_{A_n} |X| dP < \varepsilon$$

for any fixed $\varepsilon > 0$. Since ε is arbitrary, we can conclude that

$$\lim_{n\to\infty} \int_{A_n} |X| dP = 0.$$

(d) Since $0 = \mathrm{Var}(X) = E\left(X - E(X)\right)^2 = \int_{\Omega} \left(X - E(X)\right)^2 dP$, then by part (c),

$$0 = P\left(\Omega \cap \left[(X - E(X))^2 > 0\right]\right) = P\left(\Omega \cap [X \neq E(X)]\right) = P\left[X \neq E(X)\right].$$

Thus $P\left[X = E(X)\right] = 1$ so that X is equal to the constant $E(X)$ with probability 1. □

3 Exercise 5.10.15

(a) Proof:

$$0 \le nE\left(\frac{1}{X}1_{[X>n]}\right) = E\left(\frac{n}{X}1_{[X>n]}\right) \le E(1 \cdot 1_{[X>n]}) = P[X > n].$$

Since $P[0 \le X < \infty] = 1$, then

$$P[X > n] = P\left([X > n] \cap [0 \le X < \infty]\right) = P[n < X < \infty] \to 0$$

because $[n < X < \infty] \downarrow \emptyset$. Therefore,

$$\lim_{n\to\infty} nE\left(\frac{1}{X}1_{[X>n]}\right) = 0.$$

□

(b) Proof: First show that $\frac{1}{nX}1_{[X>n^{-1}]} \to 0$ as n goes to ∞. Actually, $\forall \omega \in \Omega$, if $X(\omega) > 0$, then $\frac{1}{nX(\omega)} \to 0$ as n goes to ∞; otherwise, $X(\omega) \le 0$, then $1_{[X>n^{-1}]}(\omega) = 0$ for each n.

On the other hand, $\left|\frac{1}{nX}1_{[X>n^{-1}]}\right| = \frac{1}{nX}1_{[nX>1]} \le 1$. By DCT (Dominated Convergence Theorem on page 133), $E\left(\frac{1}{nX}1_{[X>n^{-1}]}\right) \to 0$ as n goes to ∞. That is,

$$\lim_{n\to\infty} n^{-1}E\left(\frac{1}{X}1_{[X>n^{-1}]}\right) = 0.$$

□

4 Exercise 5.10.22

(a) Proof: Note that for a.e. $\omega \in \Omega$, we have $X(\omega) = X(\omega) - 0 = \int_{(0,X(\omega)]} dt$. Hence

$$\begin{aligned} E(X) &= \int_\Omega X(\omega)P(d\omega) = \int_\Omega \left(\int_{[0,X(\omega)]} dt\right) P(d\omega) \\ &= \int_\Omega \left(\int_{[0,\infty)} I_{[0,X(\omega)}(t)dt\right) P(d\omega) = \int_{[0,\infty)} \left(\int_\Omega I_{[0,X(\omega)]}(t)P(d\omega)\right) dt \\ &= \int_{[0,\infty)} P[X(\omega) > t]dt. \end{aligned}$$

The proof is completed.

(b) The proof is similar to (a) once we note that $X^\alpha(\omega) = \alpha\int_{[0,X(\omega)]} t^{\alpha-1}dt$ for a.e. $\omega \in \Omega$.

5 Exercise 5.10.28

Proof: If

$$E\left(\vee_{n=1}^{\infty}|X_n|\right) < \infty$$

simply define $Y = \vee_{n=1}^{\infty}|X_n|$.

Conversely, if there is a random variable $0 \leq Y \in L^1$, such that

$$P[|X_n| \leq Y] = 1,$$

then

$$\vee_{n=1}^{\infty}|X_n| \leq Y \quad \text{a.s..}$$

Hence,

$$E\left(\vee_{n=1}^{\infty}|X_n|\right) \leq EY < \infty.$$

The proof is completed.